JOURNAL OF COMPUTATIONAL PHYSICS **8**, 175–196 (1971)

Kalman Filtering Simulation Via Numerical Solution of the Associated Matrix Differential Equations

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Received August 18, 1970

Using the fundamental relations of Kalman's approach to optimal filtering, a digital computer simulation of the Kalman filtering process is developed. The simulation consists of numerical integration of the solution of the matrix differential equations describing the linear system random process model and the optimum filter. As a prelude to this, a brief presentation of the fundamental concepts underlying the derivation of Kalman filtering theory is given. An assumption basic to this approach is that a sufficiently accurate model of the random process can be given by a linear, possibly time-varying, dynamic system excited by white Gaussian noise. The generation, by digital techniques, of the white Gaussian noise used in the simulation, is based upon a one-dimensional variable transformation and the assumption that a uniformly distributed uncorrelated random sequence of numbers from the interval [0, 1) is available. Tests are conducted to determine the validity of this technique which is used to convert the uniformly distributed random sequence to a Gaussianly distributed random sequence and to determine the validity of the assumption that the given sequence is actually uniformly distributed and uncorrelated.

INTRODUCTION

Recently, in the field of communication theory, there has been a considerable amount of interest in estimation techniques based on the state-variable approach to representing dynamic systems. Notably among these has been the one formulated by Kalman [1–3], which involves the theory of orthogonal projection in Hilbert space. Even though this technique does not allow one to formulate problems which could not be formulated before by other well-known techniques, e.g., Wiener's theory of filtering and prediction, it does make it possible to obtain a complete solution to these problems which consists of the specification, and description of implementation, of the differential equations of the optimal filter.

In the case of conventional approaches to estimation, even when analytical results can be obtained, which is usually only for a few trivial academic examples, and certainly not when nonstationarity or time-varying behavior is considered,

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these approaches terminate with the specification of the transfer function or impulse response of the optimal filter, and therefore suffer from one major problem: the inability to describe the implementation of the optimal filter. In other words, a complete solution to the problem is not obtained since in general there is no simple method of synthesizing a filter with a prescribed transfer function or impulse response. This is the main shortcoming that Kalman's technique attempts to overcome.

With these ideas as a background, a study of the derivation of this technique, and a simulation on the digital computer of the resulting complete solutions, i.e., the filter equations, for several interesting examples were carried out. Also, a brief presentation of the fundamental concepts and ideas which were used by Kalman in his development, a description of the approach used to simulate the Kalman filter, a description of the digital techniques used to generate appropriate random noise required for the simulation, and the results obtained, are given.

Derivation of Kalman Filter Equations

Simply speaking, the Kalman filter is used to produce an estimate $\hat{\mathbf{x}}(t)$ of the values of the state-variables of a linear system which is the model of some random process and which is subject to stochastic inputs (Boldface symbols in text are equivalent to the same symbols underscored with a circumflex in figures). To do this, the Kalman filter uses as its input a signal which is a noisy observation of the output of the linear system. Then, by using an appropriate mathematical operation on this signal, it creates the desired optimal estimate. This configuration is illustrated graphically in block diagram form in Fig. 1. This is not an ordinary, but a matrix

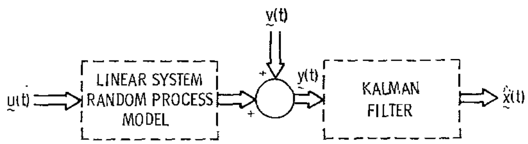

Fig. 1. Configuration of estimation problem.

block diagram as revealed by the fat lines indicating signal flow. The stochastic input to the system is an additive noise signal and is represented by $\mathbf{u}(t)$. The observation, $\mathbf{y}(t)$, is the sum of the linear system output expression which consists of a set of linear combinations of the state-variables of the system, and $\mathbf{v}(t)$, which, like $\mathbf{u}(t)$, is an additive noise signal. We assume both $\mathbf{u}(t)$ and $\mathbf{v}(t)$ to be stationary, Gaussian random processes with zero mean, and that both noise processes are white so that their correlation functions may be written as

$$E\{\mathbf{u}(t)\,\mathbf{u}^T(t+\tau)\} = \mathbf{Q}(t)\,\delta(\tau) \qquad \text{and} \qquad E\{\mathbf{v}(t)\,\mathbf{v}(t+\tau)\} = \mathbf{P}(t)\,\delta(\tau), \tag{1}$$

where $\delta(\tau)$ is the Dirac delta function. Thus, for example, the element of $\mathbf{Q}(t)$ at the intersection of the ith row and the jth column is equal to the cross-correlation function between $u_i(t)$ and $u_j(t)$, i.e.,

$$E\{u_i(t)\, u_j(t+\tau)\} = Q_{ij}(t). \tag{2}$$

Also, we assume that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are independent, thus allowing us to write

$$E\{\mathbf{u}(t)\, \mathbf{v}(t+\tau)\} = 0. \tag{3}$$

Now that the overall picture has been presented, let us state the problem formally and mathematically, and then give a brief outline of the approach we will use to obtain the form of the Kalman filter.

First, let us describe what we will call the *message* process, as the random process $\mathbf{x}(t)$ generated by the linear system model

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\, \mathbf{x}(t) + \mathbf{B}(t)\, \mathbf{u}(t), \tag{4}$$

where $\mathbf{x}(t)$ is an n-vector, $\mathbf{u}(t)$ is an m-vector ($m \leqslant n$), and $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are in general time-varying $n \times n$ and $n \times m$ matrices respectively. In other words, we characterize the message process in terms of the vector differential equations describing the linear system which would generate this process if it were excited by white noise. The observed signal is

$$\mathbf{y}(t) = \mathbf{C}(t)\, \mathbf{x}(t) + \mathbf{v}(t), \tag{5}$$

where $\mathbf{y}(t)$ and $\mathbf{v}(t)$ are p-vectors ($p \leqslant n$), and $\mathbf{C}(t)$ is a time-varying $p \times n$ matrix. The matrix product $\mathbf{C}(t)\, \mathbf{x}(t)$ is the linear system output expression. The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ have the properties described above.

Thus, the optimal estimation problem may be stated as follows: Given observed values of $\mathbf{y}(\tau)$ in the time interval $t_0 \leqslant \tau \leqslant t$, find the minimum-variance unbiased estimate $\hat{\mathbf{x}}(t_1 \mid t)$ of $\mathbf{x}(t_1)$. The notation $\hat{\mathbf{x}}(t_1 \mid t)$ represents the optimal estimate of $\mathbf{x}(t)$ at time t_1 based on all the previous observed values of $\mathbf{y}(\tau)$. Using this approach, we can consider estimation based on past data lying in either the finite or infinite time interval. Fig. 2 shows graphically the configuration of the estimation problem including the continuous-time domain form of the linear system.

Depending on whether t_1 is less than, equal to, or greater than t, the problem includes the smoothing, filtering, and prediction problems, respectively. Also, very importantly, it includes the problem of reconstructing all of the state-variables of a linear system from noisy observations of linear combinations of only *some* of the system's state-variables.

Even though we have so far postulated our problem in the continuous-time domain, and will eventually wind up with our solution postulated in the continuous-time domain, the derivation that we will present, because of inherent simplifica-

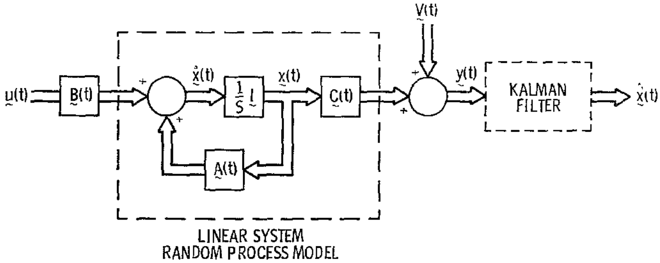

FIG. 2. Configuration of estimation problem including continuous-time domain form of linear system.

tions, will be given within the framework of the discrete-time domain. Once we obtain a solution, it is a simple straightforward manipulation to return to the continuous-time analog. In addition, we will consider only the case of discrete-time prediction at time $(t + 1)$, i.e., we will seek the optimal estimate $\hat{\mathbf{x}}(t + 1 \mid t)$ of $\mathbf{x}(t + 1)$ given all observations on $\mathbf{y}(t)$ up to time t. It is not difficult to extend these results to the cases of prediction at time $(t + n)$, $n = 1, 2, 3,...,$ and to filtering, i.e., to estimation at time t. These results may also be extended to smoothing although the derivation here is much more difficult. Since our treatment will be general enough to cover these three cases, we will refer to them collectively as estimation.

Hence, to consider our problem in the discrete-time domain, the statement of the problem will have to be modified slightly. The random process that we had before will now be represented by the discrete-time expression

$$\mathbf{x}(t + 1) = \boldsymbol{\phi}(t + 1; t)\, \mathbf{x}(t) + \mathbf{D}(t)\, \mathbf{u}(t), \tag{6}$$

where $\boldsymbol{\phi}(t + 1; t)$ is the $n \times n$ state-transition matrix of the system, $\mathbf{u}(t)$ is a white noise random sequence m-vector which is constant during each sampling period, and $\mathbf{D}(t)$ is a $n \times m$ time-varying matrix. The relationship between $\boldsymbol{\phi}(t + 1; t)$ and $\mathbf{A}(t)$, and between $\mathbf{D}(t)$ and $\mathbf{B}(t)$ will be seen later. Thus, given the observed values of $\mathbf{y}(t_0),..., \mathbf{y}(t)$, we want to find the minimum-variance unbiased estimate $\hat{\mathbf{x}}(t_1 \mid t)$ of $\mathbf{x}(t_1)$. Since we are only going to consider prediction at time $t + 1$, we replace t_1 by $t + 1$. The rest of the problem statement is essentially the same as the continuous-time case.

Before we begin our derivation, let us present a few preliminary ideas that we will need. We define the linear manifold (linear vector space) $Y(t)$ generated by $\mathbf{y}(t_0),..., \mathbf{y}(t)$ to be the set of all linear combinations

$$\sum_{i=t_0}^{t} \sum_{j=1}^{m} a_{ij}\, y_j(i) \tag{7}$$

of all m coordinates of each of the observed random vectors $\mathbf{y}(t_0),..., \mathbf{y}(t)$. We regard, abstractly, any expression of the form (7) as a "point" or "vector" in $Y(t)$.

Given any two vectors $\mathbf{a}$ and $\mathbf{b}$ in $Y(t)$, we say that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if $E\{\mathbf{ab}\} = 0$. Now, if we have any vector-valued random variable $\mathbf{x}$ [not necessarily in $Y(t)$], it can be uniquely decomposed into two parts: a part $\bar{\mathbf{x}}$ in $Y(t)$ and a part $\tilde{\mathbf{x}}$ orthogonal to $Y(t)$, i.e., orthogonal to very vector in $Y(t)$. We call $\bar{\mathbf{x}}$ the orthogonal projection of $\mathbf{x}$ on $Y(t)$ and define it as $E\{\mathbf{x}(t+1) \mid Y(t)\}$, i.e.,

$$\hat{\mathbf{x}}(t+1 \mid t) = \bar{\mathbf{x}}(t+1 \mid t) \triangleq E\{\mathbf{x}(t+1) \mid Y(t)\}. \tag{8}$$

Also we give the following definition:

$$\tilde{\mathbf{x}}(t+1 \mid t) \triangleq \mathbf{x}(t+1) - \hat{\mathbf{x}}(t+1 \mid t). \tag{9}$$

Thus, the optimal estimate, $\hat{\mathbf{x}}(t+1 \mid t)$, of $\mathbf{x}(t+1)$ given $\mathbf{y}(t_0),..., \mathbf{y}(t)$, is nothing more than the orthogonal projection of $\mathbf{x}(t+1)$ on $Y(t)$. Therefore, what we have stated is that the optimal estimate is a linear combination of all previous observations. In other words, the optimal estimate can be regarded as the output of a linear filter, with the input being the actually occurring values of the observable random variables. Hence, if we can obtain an expression for $\hat{\mathbf{x}}(t+1 \mid t)$ we will have the form of our filter.

Some further results which we state without proof, from Projection Theory in Hilbert Space, and which are basic to our treatment, are the following:

(1) If an orthogonal sequence $\mathbf{z}_1,..., \mathbf{z}_n$ generates a linear manifold, M_n, then:

(a) if $\mathbf{x} \in M_n$, $\mathbf{x} = \sum_i E\{\mathbf{x}\mathbf{z}_i^T\} E\{\mathbf{z}_i\mathbf{z}_i^T\}^{-1} \mathbf{z}_i$;

(b) if $\mathbf{x} \notin M_n$, the best (minimum-variance unbiased) estimate of $\mathbf{x}$ is $\mathbf{x} = \sum_i E\{\mathbf{x}\mathbf{z}_i^T\} E\{\mathbf{z}_i\mathbf{z}_i^T\}^{-1} \mathbf{z}_i = E\{\mathbf{x} \mid M_n\}$.

(2) If $\mathbf{z}_{n+1}$ is orthogonal to M_n, then

$$E\{\mathbf{x} \mid M_n, \mathbf{z}_{n+1}\} = E\{\mathbf{x} \mid M_n\} + E\{\mathbf{x} \mid \mathbf{z}_{n+1}\}.$$

(Notice that $E\{\mathbf{x} \mid M_n, \mathbf{z}_{n+1}\}$ is equivalent to $E\{\mathbf{x} \mid M_{n+1}\}$.)

Let us begin our derivation by assuming that $\mathbf{y}(t_0),..., \mathbf{y}(t-1)$ have been observed, i.e., that $Y(t-1)$ is known. Next, at time t, the random variable $\mathbf{y}(t)$ is observed. Let $\tilde{\mathbf{y}}(t \mid t-1)$ be the component of $\mathbf{y}(t)$ orthogonal to $Y(t-1)$. If $\tilde{\mathbf{y}}(t \mid t-1) = 0$, which means that the values of all components of this random vector are zero for almost every possible event, then $Y(t)$ is obviously the same as $Y(t-1)$ and therefore the observation of $\mathbf{y}(t)$ does not convey any additional information. This is not likely to happen in a physically meaningful situation. In any case, $\tilde{\mathbf{y}}(t \mid t-1)$ generates a linear manifold (possibly zero) which we will denote by $Z(t)$. By definition then, $Y(t-1)$ and $Z(t)$ taken together are the same manifold as $Y(t)$, and every vector in $Z(t)$ is orthogonal to every vector in $Y(t-1)$.

We shall compute $\hat{\mathbf{x}}(t+1 \mid t)$ by induction, assuming that $\mathbf{x}(t \mid t-1)$ is known. By (2) from our Projection Theory results, the conditional expectation of $\mathbf{x}(t+1)$ can be decomposed into two parts:

(i) the conditional expectation of $\mathbf{x}(t+1)$ given $Y(t-1)$, and

(ii) the conditional expectation of $\mathbf{x}(t+1)$ given $Z(t)$.

Here we recall that $Y(t-1)$ is the linear manifold generated by the observations $\mathbf{y}(t_0), \ldots, \mathbf{y}(t-1)$, and $Z(t)$ is orthogonal to $Y(t-1)$ and is the linear manifold generated by $\tilde{\mathbf{y}}(t \mid t-1)$. Thus,

$$\begin{aligned} \tilde{\mathbf{y}}(t \mid t-1) &= \mathbf{y}(t) - \bar{\mathbf{y}}(t \mid t-1) = \mathbf{y}(t) - E\{\mathbf{y}(t) \mid Y(t-1)\} \\ &= \mathbf{y}(t) - \mathbf{C}(t)\, \hat{\mathbf{x}}(t \mid t-1). \end{aligned} \tag{10}$$

In the last equation $\tilde{\mathbf{y}}(t)$ represents the new information added by making an observation of $\mathbf{y}(t)$ at time t. It is the difference between the actual observation of $\mathbf{y}(t)$ at time t, and the expected value of $\mathbf{y}(t)$ at time t based upon information available at time $t-1$.

Thus we may write the following:

$$\hat{\mathbf{x}}(t+1 \mid t) = E\{\mathbf{x}(t+1) \mid Y(t-1)\} + E\{\mathbf{x}(t+1) \mid Z(t)\}. \tag{11}$$

Since

$$\mathbf{x}(t+1) = \phi(t+1; t)\, \mathbf{x}(t) + \mathbf{D}(t)\, \mathbf{u}(t),$$

we have

$$\begin{aligned} E\{\mathbf{x}(t+1) \mid Y(t-1)\} &= E\{[\phi(t+1; t)\, \mathbf{x}(t) + \mathbf{D}(t)\, \mathbf{u}(t)] \mid Y(t-1)\} \\ &= \phi(t+1; t)\, E\{\mathbf{x}(t) \mid Y(t-1)\} + \mathbf{D}(t)\, E\{\mathbf{u}(t) \mid Y(t-1)\} \\ &= \phi(t+1; t)\, \hat{\mathbf{x}}(t \mid t-1) + \mathbf{D}(t)\, E\{\mathbf{u}(t) \mid Y(t-1)\}. \end{aligned} \tag{12}$$

Since, by assumption, $\mathbf{u}(t)$ is an independent Gaussian m-vector random process with zero mean, it follows that $\mathbf{u}(t)$ is orthogonal to $Y(t-1)$, and therefore

$$E\{\mathbf{u}(t) \mid Y(t-1)\} = 0. \tag{13}$$

Now for the second term of (11), let us use (1)-(b) of our results from Projection Theory to state that

$$\begin{aligned} &E\{\mathbf{x}(t+1) \mid Z(t)\} \\ &\quad = E\{\mathbf{x}(t+1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}\, E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1}\, \tilde{\mathbf{y}}(t \mid t-1), \end{aligned} \tag{14}$$

where the summation sign does not now appear since $Z(t)$ is the linear manifold generated by only $\mathbf{y}(t \mid t-1)$.

Next, let us see how to compute

$$E\{\mathbf{x}(t+1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} \qquad \text{and} \qquad E\{\tilde{\mathbf{y}}(t \mid t-1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1}.$$

From (6) and the fact that

$$\mathbf{x}(t) = \hat{\mathbf{x}}(t \mid t-1) + \tilde{\mathbf{x}}(t \mid t-1), \tag{15}$$

we have that

$$\begin{aligned} E\{\mathbf{x}(t+1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} \\ &= E\{[\phi(t+1;t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t)][\tilde{\mathbf{y}}^T(t \mid t-1)]\} \\ &= E\{[\phi(t+1;t)\,\tilde{\mathbf{x}}(t \mid t-1) + \phi(t+1;t)\,\hat{\mathbf{x}}(t \mid t-1) \\ &\quad + \mathbf{D}(t)\,\mathbf{u}(t)][\tilde{\mathbf{y}}^T(t \mid t-1)]\} \\ &= E\{\phi(t+1;t)\,\tilde{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} \\ &\quad + E\{\phi(t+1;t)\,\hat{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} + E\{\mathbf{D}(t)\,\mathbf{u}(t)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} \\ &= \phi(t+1;t)\,E\{\tilde{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} \\ &\quad + \phi(t+1;t)\,E\{\hat{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} + \mathbf{D}(t)\,E\{\mathbf{u}(t)\,\tilde{\mathbf{y}}^T(t \mid t-1)\}. \end{aligned} \tag{16}$$

As before, because $\mathbf{u}(t)$ is an independent Gaussian m-vector random process with zero mean, it is orthogonal to $Z(t)$ and therefore to every vector in $Z(t)$, and in particular to $\tilde{\mathbf{y}}(t \mid t-1)$. Hence,

$$E\{\mathbf{u}(t)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} = 0. \tag{17}$$

Also,

$$E\{\hat{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} = 0 \tag{18}$$

because $\hat{\mathbf{x}}(t \mid t-1)$ is a vector in $Y(t-1)$ and $\tilde{\mathbf{y}}(t \mid t-1)$ is a vector in $Z(t)$, the linear manifold which is orthogonal to $Y(t-1)$. Thus,

$$\begin{aligned} E\{\mathbf{x}(t+1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} &= \phi(t+1;t)\,E\{[\tilde{\mathbf{x}}(t \mid t-1)][\mathbf{C}(t)\,\tilde{\mathbf{x}}(t \mid t-1) + \mathbf{v}(t)]^T\} \\ &= \phi(t+1;t)\,E\{\tilde{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{x}}^T(t \mid t-1)\}\,\mathbf{C}^T(t) \\ &\quad + \phi(t+1;t)\,E\{\tilde{\mathbf{x}}(t \mid t-1)\,\mathbf{v}^T(t)\} \\ &= \phi(t+1;t)\,E\{\tilde{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{x}}^T(t \mid t-1)\}\,\mathbf{C}^T(t), \end{aligned} \tag{19}$$

since $\tilde{\mathbf{x}}(t \mid t-1)$ and $\mathbf{v}(t)$ are orthogonal. If we define

$$\mathbf{\Sigma}(t \mid t-1) = E\{\tilde{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{x}}^T(t \mid t-1)\},$$

then

$$E\{\mathbf{x}(t+1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} = \phi(t+1;t)\,\mathbf{\Sigma}(t \mid t-1)\,C^T(t). \tag{20}$$

Since

$$\tilde{\mathbf{y}}(t \mid t-1) = \mathbf{y}(t) - \hat{\mathbf{y}}(t \mid t-1) = \mathbf{y}(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t \mid t-1)$$
$$= \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{v}(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t \mid t-1) = \mathbf{C}(t)\,\tilde{\mathbf{x}}(t \mid t-1) + \mathbf{v}(t), \quad (21)$$

we may write

$$E\{\tilde{\mathbf{y}}(t \mid t-1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\}$$
$$= E\{[\mathbf{C}(t)\,\tilde{\mathbf{x}}(t \mid t-1) + \mathbf{v}(t)][\mathbf{C}(t)\,\tilde{\mathbf{x}}(t \mid t-1) + \mathbf{v}(t)]^T\}$$
$$= \mathbf{C}(t)\,E\{\tilde{\mathbf{x}}(t \mid t-1)\,\tilde{\mathbf{x}}^T(t \mid t-1)\}\,\mathbf{C}^T(t) + \mathbf{C}(t)\,E\{\tilde{\mathbf{x}}(t \mid t-1)\,\mathbf{v}^T(t)\}$$
$$+ E\{\mathbf{v}(t)\,\tilde{\mathbf{x}}^T(t \mid t-1)\}\,\mathbf{C}^T(t) + E\{\mathbf{v}(t)\,\mathbf{v}^T(t)\}. \quad (22)$$

Again,

$$E\{\tilde{\mathbf{x}}(t \mid t-1)\,\mathbf{v}^T(t)\} = 0, \quad (23)$$

and

$$E\{\mathbf{v}(t)\,\tilde{\mathbf{x}}^T(t \mid t-1)\} = 0 \quad (24)$$

because $\mathbf{v}(t)$ and $\tilde{\mathbf{x}}(t \mid t-1)$ are orthogonal. Thus,

$$E\{\tilde{\mathbf{y}}(t \mid t-1)\,\tilde{\mathbf{y}}^T(t \mid t-1)\} = \mathbf{C}(t)\,\mathbf{\Sigma}(t \mid t-1)\,\mathbf{C}^T(t) + \mathbf{P}(t). \quad (25)$$

Hence, combining equations (11), (12), (14), (20), and (25), we obtain the equation for the optimal filter:

$$\hat{\mathbf{x}}(t+1 \mid t) = \phi(t+1; t)\,\hat{\mathbf{x}}(t \mid t-1)$$
$$+ [\phi(t+1; t)\,\mathbf{\Sigma}(t \mid t-1)\,\mathbf{C}^T(t)][\mathbf{C}(t)\,\mathbf{\Sigma}(t \mid t-1)\,\mathbf{C}^T(t) + \mathbf{P}(t)]^{-1}$$
$$\times [\mathbf{y}(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t \mid t-1)]. \quad (26)$$

Let

$$\mathbf{K}(t) = [\phi(t+1; t)\,\mathbf{\Sigma}(t \mid t-1)\,\mathbf{C}^T(t)][\mathbf{C}(t)\,\mathbf{\Sigma}(t \mid t-1)\,\mathbf{C}^T(t) + \mathbf{P}(t)]^{-1}; \quad (27)$$

then

$$\hat{\mathbf{x}}(t+1 \mid t) = \phi(t+1; t)\,\hat{\mathbf{x}}(t \mid t-1) + \mathbf{K}(t)[\mathbf{y}(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t \mid t-1)]. \quad (28)$$

Of course, the initial state $\hat{\mathbf{x}}(t_0 \mid t_0 - 1) = \hat{\mathbf{x}}(t_0)$ must be specified also. This is taken to be zero since initially there are no observations and the mean of $\mathbf{x}(t_0)$ is assumed to be zero. Thus,

$$\hat{\mathbf{x}}(t_0 \mid t_0 - 1) = \hat{\mathbf{x}}(t_0) = E\{\mathbf{x}(t_0)\} = 0. \quad (29)$$

We complete the solution of the filtering problem by deriving a recursion relation for what we will define as the covariance matrix for the error between what the state-variables, $\mathbf{x}(t)$, actually are, and what we estimate them to be, $\hat{\mathbf{x}}(t)$. We designate this covariance matrix as $\boldsymbol{\Sigma}(t \mid t-1)$ and it is the only remaining unknown in (26). We will go about this in the same way that we did in deriving the recursive relation for $\hat{\mathbf{x}}(t \mid t-1)$, i.e., by computing $\boldsymbol{\Sigma}(t+1 \mid t)$ by induction assuming that $\boldsymbol{\Sigma}(t \mid t-1)$ is known. Using the definition of $\boldsymbol{\Sigma}(t+1 \mid t)$ given by

$$\boldsymbol{\Sigma}(t+1 \mid t) = E\{\tilde{\mathbf{x}}(t+1 \mid t)\, \tilde{\mathbf{x}}^T(t+1 \mid t)\}, \tag{30}$$

and parts (1)-(b) and (2) of our results from Projection Theory, we have the following:

$$\begin{aligned}\boldsymbol{\Sigma}(t+1 \mid t) &= E\{[\mathbf{x}(t+1) - \hat{\mathbf{x}}(t+1 \mid t)][\mathbf{x}(t+1) - \hat{\mathbf{x}}(t+1 \mid t)]^T\} \\ &= E\{[\mathbf{x}(t+1) - \hat{\mathbf{x}}(t+1 \mid t-1) - E\{\mathbf{x}(t+1) \mid Z(t)\}] \\ &\qquad \times [\mathbf{x}(t+1) - \hat{\mathbf{x}}(t+1 \mid t-1) - E\{x(t+1) \mid Z(t)\}]^T\} \\ &= E\{[\tilde{\mathbf{x}}(t+1 \mid t-1) - E\{\mathbf{x}(t+1) \mid Z(t)\}] \\ &\qquad \times [\tilde{\mathbf{x}}(t+1 \mid t-1) - E\{\mathbf{x}(t+1) \mid Z(t)\}]^T\},\end{aligned} \tag{31}$$

where

$$\begin{aligned}\hat{\mathbf{x}}(t+1 \mid t) = E\{\mathbf{x}(t+1) \mid Y(t)\} &= E\{\mathbf{x}(t+1) \mid Y(t-1)\} + E\{\mathbf{x}(t+1) \mid Z(t)\} \\ &= \hat{\mathbf{x}}(t+1 \mid t-1) + E\{\mathbf{x}(t+1) \mid Z(t)\}.\end{aligned} \tag{32}$$

Now

$$\begin{aligned}&E\{\mathbf{x}(t+1) \mid Z(t)\} \\ &\quad = E\{\mathbf{x}(t+1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}\, E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1}\, \tilde{\mathbf{y}}(t \mid t-1).\end{aligned} \tag{33}$$

Thus,

$$\begin{aligned}\boldsymbol{\Sigma}(t+1 \mid t) &= E\{[\tilde{\mathbf{x}}(t+1 \mid t-1) - E\{\mathbf{x}(t+1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\} \\ &\qquad \times E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1}\, \tilde{\mathbf{y}}(t \mid t-1)][\tilde{\mathbf{x}}(t+1 \mid t-1) \\ &\qquad - E\{\mathbf{x}(t+1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}\, E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1}\, \tilde{\mathbf{y}}(t \mid t-1)]^T\} \\ &= E\{\tilde{\mathbf{x}}(t+1 \mid t-1)\, \tilde{\mathbf{x}}^T(t+1 \mid t-1)\} - E\{\mathbf{x}(t+1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\} \\ &\qquad \times E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1}\, E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{x}}^T(t+1 \mid t-1)\} \\ &\qquad - E\{\tilde{\mathbf{x}}(t+1 \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}\, E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1^T} \\ &\qquad \times E\{\mathbf{x}(t+1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^T + E\{\mathbf{x}(t+1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\} \\ &\qquad \times E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1}\, E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\} \\ &\qquad \times E\{\tilde{\mathbf{y}}(t \mid t-1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^{-1^T}\, E\{\mathbf{x}(t+1)\, \tilde{\mathbf{y}}^T(t \mid t-1)\}^T.\end{aligned} \tag{34}$$

Notice that

$$E\{\mathbf{x}(t+1)\,\tilde{\mathbf{y}}^T(t\mid t-1)\} = E\{[\hat{\mathbf{x}}(t+1\mid t-1) + \tilde{\mathbf{x}}(t+1\mid t-1)][\tilde{\mathbf{y}}^T(t\mid t-1)]\}$$
$$= E\{\tilde{\mathbf{x}}(t+1\mid t-1)\,\tilde{\mathbf{y}}^T(t\mid t-1)\} \tag{35}$$

since $\hat{\mathbf{x}}(t+1\mid t-1)$ and $\tilde{\mathbf{y}}(t\mid t-1)$ are orthogonal. Thus simplifying (34) gives

$$\boldsymbol{\Sigma}(t+1\mid t) = E\{\tilde{\mathbf{x}}(t+1\mid t-1)\,\tilde{\mathbf{x}}^T(t+1\mid t-1)\} - E\{\mathbf{x}(t+1)\,\tilde{\mathbf{y}}^T(t\mid t-1)\}$$
$$\times E\{\tilde{\mathbf{y}}(t\mid t-1)\,\tilde{\mathbf{y}}^T(t\mid t-1)\}^{-1}\,E\{\mathbf{x}(t+1)\,\tilde{\mathbf{y}}^T(t\mid t-1)\}^T. \tag{36}$$

Now using the relation

$$\tilde{\mathbf{x}}(t+1\mid t-1) = \mathbf{x}(t+1) - \hat{\mathbf{x}}(t+1\mid t-1)$$
$$= \boldsymbol{\phi}(t+1;t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t) - \boldsymbol{\phi}(t+1;t)\,\hat{\mathbf{x}}(t\mid t-1)$$
$$= \boldsymbol{\phi}(t+1;t)\,\tilde{\mathbf{x}}(t\mid t-1) + \mathbf{D}(t)\,\mathbf{u}(t), \tag{37}$$

we get

$$E\{\tilde{\mathbf{x}}(t+1\mid t-1)\,\tilde{\mathbf{x}}^T(t+1\mid t-1)\}$$
$$= E\{[\boldsymbol{\phi}(t+1;t)\,\tilde{\mathbf{x}}(t\mid t-1) + \mathbf{D}(t)\,\mathbf{u}(t)][\boldsymbol{\phi}(t+1;t)\,\tilde{\mathbf{x}}(t\mid t-1) + \mathbf{D}(t)\,\mathbf{u}(t)]^T\}$$
$$= E\{\boldsymbol{\phi}(t+1;t)\,\tilde{\mathbf{x}}(t\mid t-1)\,\tilde{\mathbf{x}}^T(t\mid t-1)\,\boldsymbol{\phi}^T(t+1;t)\}$$
$$+ E\{\boldsymbol{\phi}(t+1;t)\,\tilde{\mathbf{x}}(t\mid t-1)\,\mathbf{u}^T(t)\,\mathbf{D}^T(t)\}$$
$$+ E\{\mathbf{D}(t)\,\mathbf{u}(t)\,\tilde{\mathbf{x}}^T(t\mid t-1)\,\boldsymbol{\phi}^T(t+1;t)\} + E\{\mathbf{D}(t)\,\mathbf{u}(t)\,\mathbf{u}^T(t)\,\mathbf{D}^T(t))\}$$
$$= \boldsymbol{\phi}(t+1;t)\,E\{\tilde{\mathbf{x}}(t\mid t-1)\,\tilde{\mathbf{x}}^T(t\mid t-1)\}\,\boldsymbol{\phi}^T(t+1;t)$$
$$+ \boldsymbol{\phi}(t+1;t)\,E\{\tilde{\mathbf{x}}(t\mid t-1)\,\mathbf{u}^T(t)\}\,\mathbf{D}^T(t) + \mathbf{D}(t)\,E\{\mathbf{u}(t)\,\tilde{\mathbf{x}}^T(t\mid t-1)\}$$
$$\times \boldsymbol{\phi}^T(t+1;t) + \mathbf{D}(t)\,E\{\mathbf{u}(t)\,\mathbf{u}^T(t)\}\,\mathbf{D}^T(t), \tag{38}$$

or

$$E\{\tilde{\mathbf{x}}(t+1\mid t-1)\,\tilde{\mathbf{x}}^T(t+1\mid t-1)\}$$
$$= \boldsymbol{\phi}(t+1;t)\,\boldsymbol{\Sigma}(t\mid t-1)\,\boldsymbol{\phi}^T(t+1;t) + \mathbf{D}(t)\,\mathbf{Q}(t)\,\mathbf{D}^T(t). \tag{39}$$

Substituting equations (20), (25), and (39) into (36) we get

$$\boldsymbol{\Sigma}(t+1\mid t) = \boldsymbol{\phi}(t+1;t)\,\boldsymbol{\Sigma}(t\mid t-1)\,\boldsymbol{\phi}^T(t+1;t) + \mathbf{D}(t)\,\mathbf{Q}(t)\,\mathbf{D}^T(t)$$
$$- [\boldsymbol{\phi}(t+1;t)\,\boldsymbol{\Sigma}(t\mid t-1)\,\mathbf{C}^T(t)][\mathbf{C}(t)\,\boldsymbol{\Sigma}(t\mid t-1)\,\mathbf{C}^T(t) + \mathbf{P}(t)]^{-1}$$
$$\times [\boldsymbol{\phi}(t+1;t)\,\boldsymbol{\Sigma}(t\mid t-1)\,\mathbf{C}^T(t)]^T. \tag{40}$$

We shall call equation (40) the *Variance Equation*. Two features of this equation are noteworthy. First, the equation does not involve the observations $\mathbf{y}(t)$. Since the gains of the optimal filter are governed by the variance equation, this means that the structure of the optimal filter (i.e., the element values) can be determined independently of the random data $\mathbf{y}(t)$.

Second, equations (26) and (40) together completely determine the conditional distribution of the random sequence for $t_1 = t + 1$, given $\mathbf{y}(t_0),..., \mathbf{y}(t)$. In other [redacted]

$\boldsymbol{\Sigma}(t_0 \mid t_0 - 1)$ is given. This is regarded as part of the problem statement and is defined as follows:

$$\boldsymbol{\Sigma}(t_0 \mid t_0 - 1) = \boldsymbol{\Sigma}(t_0) = \text{cov}[\mathbf{x}(t_0)] = E\{[\mathbf{x}(t_0) - E\{\mathbf{x}(t_0)\}][\mathbf{x}(t_0) - E\{\mathbf{x}(t_0)\}]^T\}. \quad (41)$$

This definition is a result of having selected the initial state of the estimate of the state-variables as

$$\hat{\mathbf{x}}(t_0 \mid t_0 - 1) = \hat{\mathbf{x}}(t_0) = E\{\mathbf{x}(t_0)\}. \quad (42)$$

At this point we have completely defined the problem. Let us therefore review for a moment and rewrite the important relationships.

(a) The linear system random sequence model which generates our message process $x(t)$ is completely characterized by the equations

$$\begin{aligned} \mathbf{x}(t+1) &= \boldsymbol{\phi}(t+1; t)\, \mathbf{x}(t) + \mathbf{D}(t)\, \mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}(t)\, \mathbf{x}(t) + \mathbf{v}(t). \end{aligned} \quad (43)$$

(b) The characteristics of the noise are described by the equations

$$\begin{aligned} &E\{\mathbf{u}(t)\, \mathbf{u}^T(t+\tau)\} = \mathbf{Q}(t)\, \delta(\tau), \qquad E\{\mathbf{u}(t)\} = 0, \\ &E\{\mathbf{v}(t)\, \mathbf{v}^T(t+\tau)\} = \mathbf{P}(t)\, \delta(\tau), \qquad E\{\mathbf{v}(t)\} = 0, \\ &E\{\mathbf{u}(t)\, \mathbf{v}^T(t+\tau)\} = 0, \end{aligned} \quad (44)$$

where δ is the Dirac delta function, and $\mathbf{Q}(t)$ and $\mathbf{P}(t)$ are symmetric matrices.

(c) The initial conditions are given by

$$\hat{\mathbf{x}}(t_0 \mid t_0 - 1) = \hat{\mathbf{x}}(t_0) = E\{\mathbf{x}(t_0)\} = 0,$$

$$\boldsymbol{\Sigma}(t_0 \mid t_0 - 1) = \boldsymbol{\Sigma}(t_0) = \text{cov}[\mathbf{x}(t_0)] = E\{[\mathbf{x}(t_0) - E\{\mathbf{x}(t_0)\}][\mathbf{x}(t_0) - E\{\mathbf{x}(t_0)\}]^T\}.$$

(d) The optimal filter is completely characterized by the following three equations:

(i) $\hat{\mathbf{x}}(t+1 \mid t) = \boldsymbol{\phi}(t+1; t)\, \hat{\mathbf{x}}(t \mid t-1) + \mathbf{K}(t)[\mathbf{y}(t) - \mathbf{C}(t)\, \hat{\mathbf{x}}(t \mid t-1)]$;

(ii) $\mathbf{K}(t) = [\boldsymbol{\phi}(t+1; t)\, \boldsymbol{\Sigma}(t \mid t-1)\, \mathbf{C}^T(t)][\mathbf{C}(t)\, \boldsymbol{\Sigma}(t \mid t-1)\, \mathbf{C}^T(t) + \mathbf{P}(t)]^{-1}$, where $\boldsymbol{\Sigma}(t \mid t-1) = E\{\tilde{\mathbf{x}}(t \mid t-1)\, \tilde{\mathbf{x}}^T(t \mid t-1)\}$;

(iii) $\boldsymbol{\Sigma}(t+1 \mid t) = \phi(t+1; t)\, \boldsymbol{\Sigma}(t \mid t-1)\, \phi^T(t+1; t) + \mathbf{D}(t)\, \mathbf{Q}(t)\, \mathbf{D}^T(t) - [\phi(t+1; t)\, \boldsymbol{\Sigma}(t \mid t-1)\, \mathbf{C}^T(t)][\mathbf{C}(t)\, \boldsymbol{\Sigma}(t \mid t-1)\, \mathbf{C}^T(t) + \mathbf{P}(t)]^{-1}[\phi(t+1; t) \times \boldsymbol{\Sigma}(t \mid t-1)\, \mathbf{C}^T(t)]^T$,

where the last equation is the *Variance Equation.*

Thus, we have developed the recursive relations to iteratively evaluate the optimal Kalman filtering process. Equations (26) and (40) describe the structure of the filter and the general block diagram is shown in Fig. 3. It is a feedback system built around our model of the random sequence described by Eq. (43). The error signal $\tilde{\mathbf{y}}(t \mid t-1)$ is fed forward into the model with gain $\mathbf{K}(t)$. The gain is such that the input to the model is the conditional expectation of $\mathbf{x}(t+1)$ given the observed difference $\mathbf{y}(t) - \hat{\mathbf{y}}(t \mid t-1)$, i.e., the magnitude of $\mathbf{K}(t)$ is indicative of the amount of information contained in the signal $\tilde{\mathbf{y}}(t \mid t-1)$ about the state $\mathbf{x}(t+1)$.

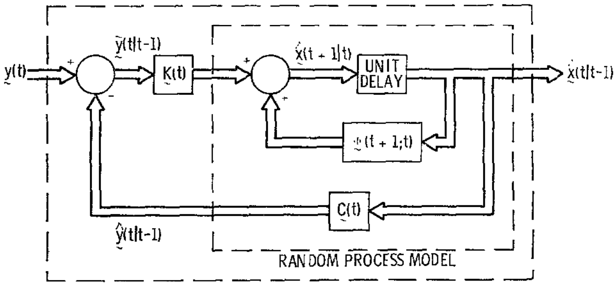

FIG. 3. Discrete-time domain structure of Kalman filter.

Now, as was stated earlier, by allowing the sampling period (equal to unity so far) to approach zero in the limit, it is possible to obtain expressions describing the optimal filtering process in the continuous-time domain. The derivation which is used to do this is semirigorous and will be described only briefly.

Let q be a positive integer and let the time t be discrete so that its successive values differ by q^{-1}. Then, assuming that $\phi(t+1; t)$ is the transition matrix of a continuous-time linear dynamic system, we have

$$\phi(t+q^{-1}; t) = \mathbf{I} + q^{-1}\mathbf{A}(t) + \mathbf{0}(q^{-1}),$$

$$\mathbf{D}(t) = \int_t^{t+q^{-1}} \phi(t+1; \tau)\, \mathbf{B}(\tau)\, d\tau = q^{-1}\mathbf{B}(t) + \mathbf{0}(q^{-1}), \tag{45}$$

where $\mathbf{0}(q^{-1})$ denotes a matrix which is zero in the limit as $q \to \infty$. Next, guided by procedures (which we will not go into here) about how to define Gaussian white noise processes as the formal limit of appropriate Gaussian white noise sequences [3], the covariance matrices $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ are replaced by $q\mathbf{P}(t)$ and $q\mathbf{Q}(t)$, respectively.

Substituting these expressions into Eq. (6), we obtain

$$\begin{aligned}\frac{\mathbf{x}(t+q^{-1})-\mathbf{x}(t)}{q^{-1}} &= \frac{1}{q^{-1}}\{[\mathbf{I}+q^{-1}\mathbf{A}(t)+\mathbf{0}(q^{-1})]\,\mathbf{x}(t)+[q^{-1}\mathbf{B}(t)+\mathbf{0}(q^{-1})]\,\mathbf{u}(t)-\mathbf{x}(t)\}\\ &= \mathbf{A}(t)\,\mathbf{x}(t)+\mathbf{B}(t)\,\mathbf{u}(t)+\mathbf{0}(q^{-1}).\end{aligned}$$

Thus, in the limit we have

$$\mathring{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t)+\mathbf{B}(t)\,\mathbf{u}(t).$$

From Eq. (40) we see that

$$\begin{aligned}&\frac{\mathbf{\Sigma}(t+q^{-1}\mid t)-\mathbf{\Sigma}(t\mid t-q^{-1})}{q^{-1}}\\ &= \frac{1}{q^{-1}}\{[\mathbf{I}+q^{-1}\mathbf{A}(t)+\mathbf{0}(q^{-1})]\,\mathbf{\Sigma}(t\mid t-q^{-1})[\mathbf{I}+q^{-1}\mathbf{A}(t)+\mathbf{0}(q^{-1})]^T\\ &\quad + q^{-1}\mathbf{B}(t)\,q\mathbf{Q}(t)\,q^{-1}\mathbf{B}^T(t)-\{[\mathbf{I}+q^{-1}\mathbf{A}(t)+\mathbf{0}(q^{-1})]\,\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)\}\\ &\quad \times\{\mathbf{C}(t)\,\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)+q\mathbf{P}(t)\}^{-1}\{[\mathbf{I}+q^{-1}\mathbf{A}(t)+\mathbf{0}(q^{-1})]\\ &\quad \times\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)\}^T-\mathbf{\Sigma}(t\mid t-q^{-1})\}\\ &= \frac{1}{q^{-1}}\{q^{-1}[\mathbf{A}(t)\,\mathbf{\Sigma}(t\mid t-q^{-1})+\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{A}^T(t)]+q^{-1}\mathbf{B}(t)\,\mathbf{Q}(t)\,\mathbf{B}^T(t)\\ &\quad -\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)[q^{-1}\mathbf{C}(t)\,\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)+\mathbf{P}(t)]^{-1}\,q^{-1}\mathbf{C}(t)\\ &\quad \times\mathbf{\Sigma}(t\mid t-q^{-1})-q^{-1}\mathbf{A}(t)\,\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)[q^{-1}\mathbf{C}(t)\,\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)\\ &\quad +\mathbf{P}(t)]^{-1}\,q^{-1}\mathbf{C}(t)\,\mathbf{\Sigma}^T(t\mid t-q^{-1})-\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)[q^{-1}\mathbf{C}(t)\\ &\quad \times\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)+\mathbf{P}(t)]^{-1}\,q^{-1}\mathbf{C}(t)\,\mathbf{\Sigma}^T(t\mid t-q^{-1})\,q^{-1}\mathbf{A}^T(t)-q^{-1}\mathbf{A}(t)\\ &\quad \times\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)[q^{-1}\mathbf{C}(t)\,\mathbf{\Sigma}(t\mid t-q^{-1})\,\mathbf{C}^T(t)+\mathbf{P}(t)]^{-1}\,q^{-1}\mathbf{C}(t)\\ &\quad \times\mathbf{\Sigma}^T(t\mid t-q^{-1})\,q^{-1}\mathbf{A}^T(t)+\mathbf{0}(q^{-1})\}.\end{aligned}$$

In the limit then, the fourth, fifth, and sixth terms in the above equation vanish, and

$$\begin{aligned}\mathring{\mathbf{\Sigma}}(t\mid t) &= \mathbf{A}(t)\,\mathbf{\Sigma}(t\mid t)+\mathbf{\Sigma}(t\mid t)\,\mathbf{A}^{\iota}(t)+\mathbf{B}(t)\,\mathbf{Q}(t)\,\mathbf{B}^T(t)\\ &\quad -\mathbf{\Sigma}(t\mid t)\,\mathbf{C}^T(t)\,\mathbf{P}^{-1}(t)\,\mathbf{C}(t)\,\mathbf{\Sigma}^T(t\mid t).\end{aligned}$$

If we let $\mathring{\mathbf{R}}(t) = \mathbf{\Sigma}(t\mid t)$ we obtain

$$\mathring{\mathbf{R}}(t) = \mathbf{A}(t)\,\mathbf{R}(t)+\mathbf{R}(t)\,\mathbf{A}^T(t)+\mathbf{B}(t)\,\mathbf{Q}(t)\,\mathbf{B}^T(t)-\mathbf{R}(t)\,\mathbf{C}^T(t)\,\mathbf{P}^{-1}(t)\,\mathbf{C}(t)\,\mathbf{R}^T(t). \quad (46)$$

From Eq. (28) we have

$$\frac{\hat{\mathbf{x}}(t + q^{-1} \mid t) - \hat{\mathbf{x}}(t \mid t - q^{-1})}{q^{-1}} = \frac{1}{q^{-1}} \{[\mathbf{I} + q^{-1}\mathbf{A}(t) + \mathbf{0}(q^{-1})]\, \hat{\mathbf{x}}(t \mid t - q^{-1}) + \mathbf{K}(t)[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t \mid t - q^{-1})] - \mathbf{x}(t \mid t - q^{-1})\}$$
$$= \mathbf{A}(t)\, \hat{\mathbf{x}}(t \mid t - q^{-1}) + q\mathbf{K}(t)[\mathbf{y}(t) - \mathbf{C}(t)\, \hat{\mathbf{x}}(t \mid t - q^{-1})] + \mathbf{0}(q^{-1}). \tag{47}$$

Now let us stop for a moment and consider the expression for $q\mathbf{K}(t)$. From Eq. (27)

$$q\mathbf{K}(t) = q\{[\mathbf{I} + q^{-1}\mathbf{A}(t) + \mathbf{0}(q^{-1})]\, \mathbf{\Sigma}(t \mid t - q^{-1})\, \mathbf{C}^T(t)\} \times \{\mathbf{C}(t)\, \mathbf{\Sigma}(t \mid t - q^{-1})\, \mathbf{C}^T(t) + q\mathbf{P}(t)\}^{-1}$$
$$= \mathbf{\Sigma}(t \mid t - q^{-1})\, \mathbf{C}(t)[q^{-1}\mathbf{C}(t)\, \mathbf{\Sigma}(t \mid t - q^{-1})\, \mathbf{C}^T(t) + \mathbf{P}(t)]^{-1} + \mathbf{A}(t)\, \mathbf{\Sigma}(t \mid t - q^{-1})\, \mathbf{C}^T(t) \times [q^{-1}\mathbf{C}(t)\, \mathbf{\Sigma}(t \mid t - q^{-1})\, \mathbf{C}^T(t) + \mathbf{P}(t)]^{-1} q^{-1}.$$

Passing to the limit then, $q\mathbf{K}(t)$ becomes $\bar{\mathbf{K}}(t)$, where

$$\bar{\mathbf{K}}(t) = \mathbf{\Sigma}(t \mid t)\, \mathbf{C}^T(t)\, \mathbf{P}^{-1}(t) = \mathbf{R}(t)\, \mathbf{C}^T(t)\, \mathbf{P}^{-1}(t).$$

Hence, Eq. (47) becomes

$$\mathring{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\, \hat{\mathbf{x}}(t) + \bar{\mathbf{K}}(t)[\mathbf{y}(t) - \mathbf{C}(t)\, \hat{\mathbf{x}}(t)]. \tag{48}$$

Thus we may replace the equations for our discrete-time problem by the following equations for the continuous-time case.

(a) The linear system random process model is now completely characterized by the equations

$$\begin{aligned} \mathring{\mathbf{x}}(t) &= \mathbf{A}(t)\, \mathbf{x}(t) + \mathbf{B}(t)\, \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\, \mathbf{x}(t) + \mathbf{v}(t). \end{aligned} \tag{49}$$

(b) The optimal filter is now characterized by the equations

$$\begin{aligned} \mathring{\hat{\mathbf{x}}}(t) &= \mathbf{A}(t)\, \hat{\mathbf{x}}(t) + \bar{\mathbf{K}}(t)[\mathbf{y}(t) - \mathbf{C}(t)\, \hat{\mathbf{x}}(t)] \\ \bar{\mathbf{K}}(t) &= \mathbf{R}(t)\, \mathbf{C}^T(t)\, \mathbf{P}^{-1}(t) \\ \mathring{\mathbf{R}}(t) &= \mathbf{A}(t)\, \mathbf{R}(t) + \mathbf{R}(t)\, \mathbf{A}^T(t) + \mathbf{B}(t)\, \mathbf{Q}(t)\, \mathbf{B}^T(t) - \mathbf{R}(t)\, \mathbf{C}^T(t)\, \mathbf{P}^{-1}(t)\, \mathbf{C}(t)\, \mathbf{R}^T(t). \end{aligned} \tag{50}$$

The descriptions of the noise and initial conditions are essentially the same as before.

Equations (49) and (50) represent the Kalman filtering problem in the continuous-time domain and are the equations used for the digital-computer simulation. In this simulation, the initial condition for the estimate of the system state-variables was always selected to be zero. The noise was generated digitally and was assumed to have the appropriate characteristics. More is said about this noise generation problem in a later section.

One interesting variation on the basic Kalman filtering problem which was tried without rigorous derivation, but with interesting results, was the following:

$$\text{(a)}\quad \begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)[\mathbf{g}(t) + \mathbf{u}(t)], \\ \mathbf{y}(t) &= \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{v}(t), \end{aligned} \tag{51}$$

$$\text{(b)}\quad \begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}(t)\,\hat{\mathbf{x}}(t) + \mathbf{B}(t)\,\mathbf{g}(t) + \bar{\mathbf{K}}(t)[\mathbf{y}(t) - \mathbf{C}(t)\,\mathbf{x}(t)] \\ \bar{\mathbf{K}}(t) &= \mathbf{R}(t)\,\mathbf{C}^T(t)\,\mathbf{P}^{-1}(t) \\ \dot{\mathbf{R}}(t) &= \mathbf{A}(t)\,\mathbf{R}(t) + \mathbf{R}(t)\,\mathbf{A}^T(t) + \mathbf{B}(t)\,\mathbf{Q}(t)\,\mathbf{B}^T(t) \\ &\quad - \mathbf{R}(t)\,\mathbf{C}^T(t)\,\mathbf{P}^{-1}(t)\,\mathbf{C}(t)\,\mathbf{R}^T(t), \end{aligned} \tag{52}$$

where $\mathbf{g}(t)$ represents some desired input to the system, such as an optimal control vector.

Digital Computer Simulation of Kalman Filter

The first configuration which was digitally simulated was that described by Eqs. (49) and (50). The second configuration which was simulated is described by Eqs. (51) and (52). Figures 4 through 9 show the simulation results for the two different systems. The describing matrices and noise properties are shown for each system.

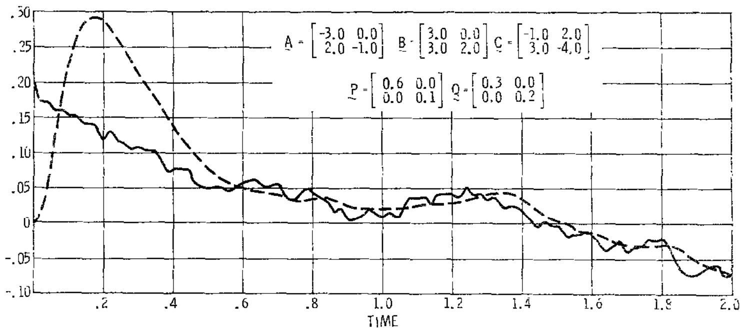

FIG. 4. State variable $X_1(t)$ of linear system random process model (——), and its optimal Kalman filter estimate (– – – –).

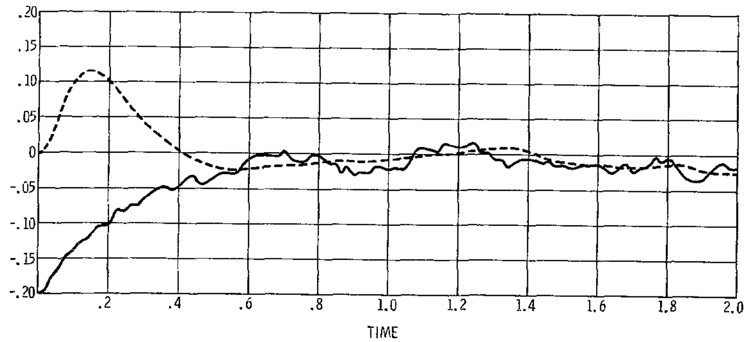

FIG. 5. State variable $X_2(t)$ of linear system random process model (——), and its optimal Kalman filter estimate (– – – –).

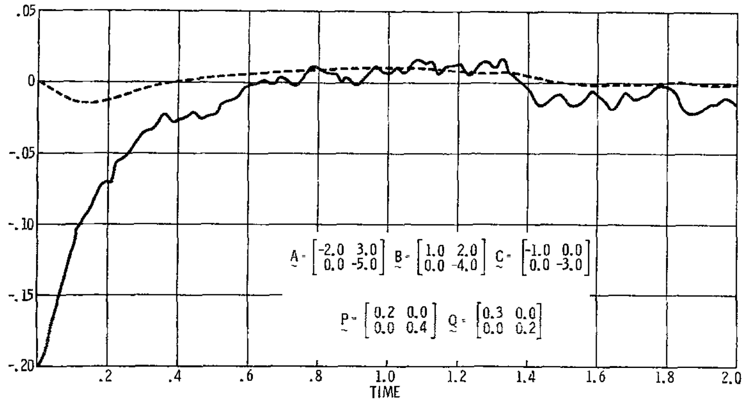

FIG. 6. State variable $X_1(t)$ of linear system random process model (——), and its optimal Kalman filter estimate (– – – –).

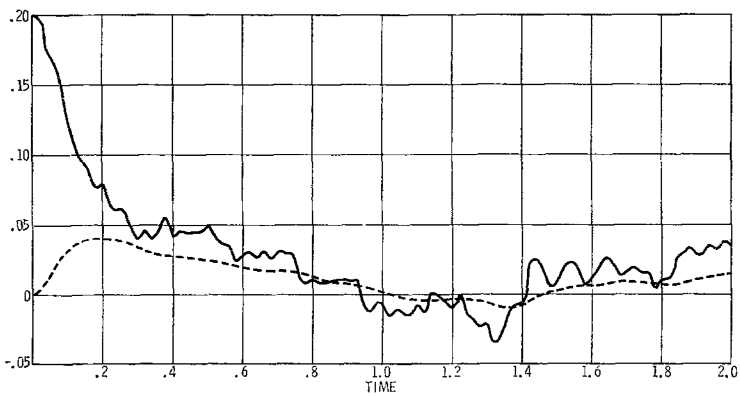

FIG. 7. State variable $X_2(t)$ of linear system random process model (——), and its optimal

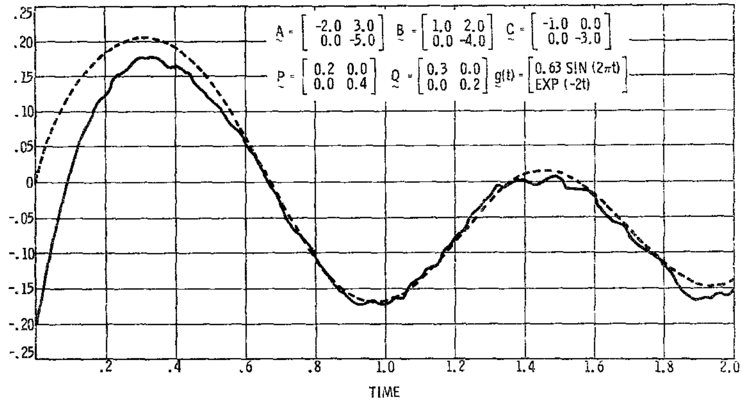

FIG. 8. State variable $X_1(t)$ of linear system random process model (——), and its optimal Kalman filter estimate (– – – –).

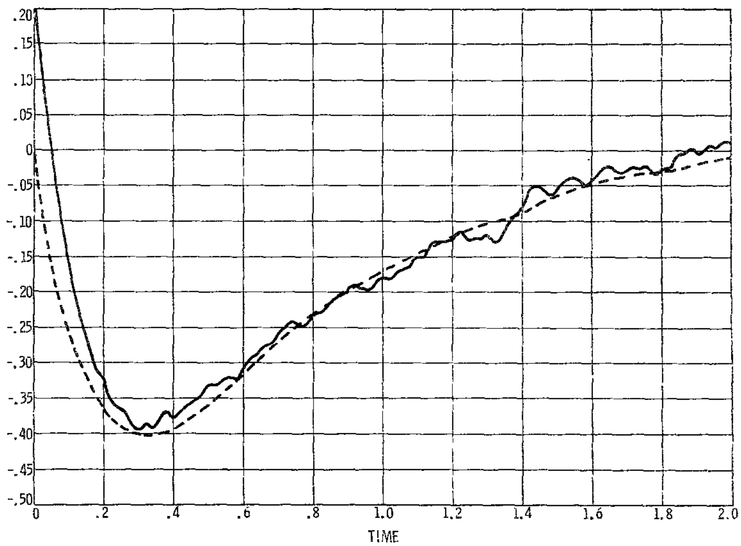

FIG. 9. State variable $X_2(t)$ of linear system random process model (——), and its optimal Kalman filter estimate (– – – –).

METHOD OF NOISE GENERATION AND TESTS FOR RANDOMNESS

The method of generating random noise was based on the idea that given a random sequence of numbers which appear to be drawn from a uniform distribution, an appropriate one-dimensional variable transformation may be made to produce a subsequent random sequence of numbers which appear to be drawn from a normal, or Gaussian, distribution. A congruential method of generating random numbers was used to obtain a uniformly distributed random sequence. This method will not be described here since it is well known and is described fully in the literature. However, tests for how closely the random sequences approached uniformity were carried out, and will be described subsequently.

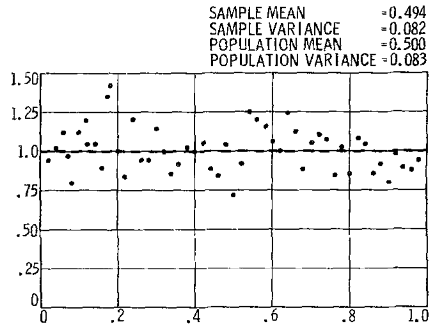

FIG. 10. Distribution of approximately uniformly distributed random sequence (·), and true uniform distribution (– – – –).

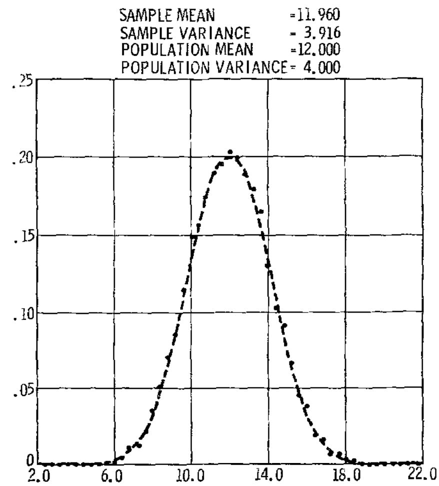

FIG. 11. Distribution of approximately Gaussianly distributed random sequence (·), and true Gaussian distribution (– – – –).

Figure 10 shows the distribution of a random sequence of 1000 numbers generated by a congruential method which was designed to produce approximately uniformly distributed random numbers in the interval [0, 1). This distribution was obtained by dividing [0, 1) into equal subintervals and counting the numbers that fell into each subinterval. Also plotted in the same figure is the true uniform distribution. Figure 11 is a similar plot for a sequence of 1000 random numbers generated by transforming by standard means a uniformly distributed random sequence of numbers. Also plotted in the same figure is the true Gaussian distribution.

Thus we see that if we have a uniformly distributed random sequence, it can be transformed into a Gaussianly distributed random sequence. The question that remains, however, is whether or not the given random sequence is acceptably uniformly distributed.

To test the sequence of random numbers for this property, the interval over which these numbers are distributed, [0, 1), can be divided into k equal subintervals. Then the frequency f_i can be determined for a sequence of n numbers, where f_i is the number of the numbers in the i-th interval, and the statistic

$$\chi_1^2 = \frac{k}{n} \sum_{i=1}^{k} \left(f_i - \frac{n}{k}\right)^2 \tag{53}$$

can be computed. It is well known [4] that this statistic has a chi-squared distribution with $k - 1$ degrees of freedom provided that n is large and the sequence is actually drawn at random from the uniform distribution. Thus, for example, if $k = 10$, this statistic should not exceed 16.9 more than 5% of the time.

This χ_1^2 statistic was calculated for a block of 1000 numbers in a sequence, repeated with a second block, and so on, for 100 blocks. The 100 values of χ_1^2 which were obtained in this way, and 100 corresponding values from a true chi-square probability density function, were plotted to see how closely the two matched. In addition, the first sample moment about zero, m_1', and the second sample moment, m_2, about the expected value of the first sample moment, were calculated to compare with the mean and variance of a true chi-square distribution with $k - 1$ degrees of freedom. We may express these sample moments as

$$m_1' = \frac{1}{n} \sum_{i=1}^{n} \chi_{1_i}^2, \tag{54}$$

and

$$m_2 = \frac{1}{n} \sum_{i=1}^{n} \{\chi_{1_i}^2 - E[m_1']\}^2. \tag{55}$$

Since the expected value of the r-th sample moment about zero is equal to the r-th population moment about zero, i.e., $E[m_r'] = \mu_r'$, we have

$$E[m_1'] = \mu_1',$$

and

$$E[m_2] = E\left\{\frac{1}{n}\sum_{i=1}^{n}\{\chi_{1_i}^2 - E[m_1']\}^2\right\} = E\left\{\frac{1}{n}\sum_{i=1}^{n}\{\chi_{1_i}^2 - \mu_1'\}^2\right\} = \mu_2' - [\mu_1']^2. \quad (56)$$

For the chi-square distribution with r degrees of freedom,

$$\mu_1' = r \quad \text{and} \quad \mu_2' = r(r+2). \quad (57)$$

Therefore,

$$E[m_1'] = r \quad \text{and} \quad E[m_2] = 2r. \quad (58)$$

For the case considered here, the interval $[0, 1)$ was divided into $k = 10$ subintervals. Thus the number of degrees of freedom was equal to 9, and therefore,

$$E[m_1'] = 9, \quad \text{and} \quad E[m_2] = 18. \quad (59)$$

Figure 12 shows a sample distribution obtained as just described, and also a plot of a true chi-square distribution with the same number, i.e., 9, degrees of freedom. The sample moments and corresponding population moments are also shown.

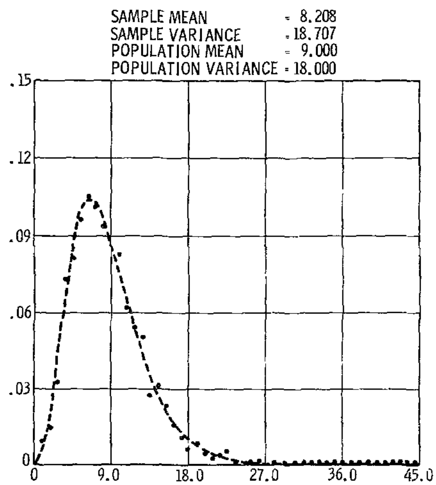

FIG. 12. Distribution of χ_1^2 statistic and true chi-square distribution with 9 degrees of freedom.

In addition to being uniformly distributed, another requirement of a random sequence that was of interest here was that there be no correlation between each number and the one immediately following it in the sequence. One way to test for this property is to determine the frequency f_{ij} for the sequence, where f_{ij} is the number of numbers in the i-th interval which are followed by a number in the j-th interval. Then the statistic

$$\chi_2^2 = \frac{k^2}{n} \sum_{i,j=1}^{k} \left(f_{ij} - \frac{n}{k^2}\right)^2 \tag{60}$$

can be computed, where k is again the number of subintervals into which the interval [0, 1) is divided. It was shown by Good [5] that $\chi_2^2 - \chi_1^2$ has an asymptotically chi-squared distribution with $k^2 - k$ degrees of freedom. This statistic was calculated for 100 blocks of 1000-number sequences just as in the previous test. The rest of the test was carried out just as before, but here the number of degrees of freedom was 90, with the result that

$$E[m_1'] = 90 \qquad \text{and} \qquad E[m_2] = 180. \tag{61}$$

Figure 13 shows a sample distribution and a plot of a true chi-square distribution with 90 degrees of freedom. The sample moments and the corresponding population moments are also shown.

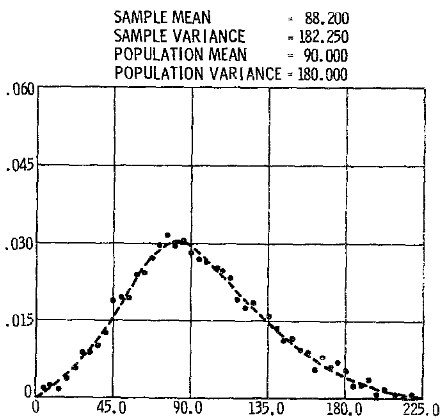

FIG. 13. Distribution of $\chi_2^2 - \chi_1^2$ statistic and true chi-square distribution with 90 degrees of freedom.

Summary and Conclusions

A brief presentation of the fundamental concepts underlying the derivation of Kalman filtering theory has been given. This problem may be described most concisely as follows: Given noisy observations of past data lying in the finite or infinite time-interval on the output of a linear system which is the model of some random process, and which is excited by white Gaussian noise, we seek the best linear estimate of the state of this linear system. An assumption basic to this approach is that a sufficiently accurate model of the random process can be given by a linear, possibly time-varying, dynamic system excited by white Gaussian noise.

The fundamental relations of Kalman's approach as considered here consist of four equations:

(i) The differential equations governing the optimal filter which is excited by the observed signals and generates the best linear estimate of the state of the model of the random process.

(ii) The differential equations governing the error of the best linear estimate.

(iii) The time-varying gains of the optimal filter expressed in terms of the error variances.

(iv) The nonlinear differential equation governing the covariance matrix of the errors of the best linear estimate, called the *Variance Equation*.

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